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ON THE HYPERBOLIC VARIATION OF ELASTIC MODULUS IN A NON-HOMOGENEOUS STRATUM

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Abstract—An exact formulation is presented for the governing dual integral equations representing the mixed boundary-value problem of the static stress distribution under a long rigid rectangular body lying on the free surface of a non-homogeneous stratum. The shear modulus of the stratum increases in the depth direction y from a value G_0 at the surface according to the hyperbolic variation.

$$G(y)=\frac{G_0h}{h-y}$$

It, therefore, simulates a practical soil, of arbitrary Poisson's ratio, which smoothly merges into a rigid bed at a depth h below the surface. The work shows that the problem is governed by both kinds of exponential integral function and that the effect of surface elastic properties is dominant in the solution of the governing equations. The limiting case of a homogeneous half-space is easily recovered from the general formulation as h tends to infinity.

1. INTRODUCTION

Attempts have been made recently to break through the difficulties of analysing the behaviour of practical elastic media subjected to surface loading by removing the usual assumption of a homogeneous continuum. Smooth inhomogeneity due to a linearly increasing modulus of elasticity with depth was first introduced by Gibson [1] for a half-space and by Gibson *et al.* [2] for a stratum.

Except in the torsional mode of vibration which is independent of Poisson's ratio, Awojobi [3] or the half-space problem, Awojobi and Gibson [4], the analytical details in the case of smooth inhomogeneity have constrained the authors to assume that the medium is incompressible. The stratum problem for a compressible medium with linearly increasing shear modulus is, therefore, yet to be considered. However, a practical variation of modulus with depth which does not require the assumption of an incompressible medium would throw some light on the hypotheses that the surface modulus is the dominant parameter for problems of any non-homogeneous medium in which lateral variation of elastic properties is negligible. This theory has been established by the author in recent works dealing with the linearly increasing modulus media.

The present investigation is concerned with the static stress and displacement fields in a non-homogeneous stratum of arbitrary Poisson's ratio in which the shear modulus varies according to the hyperbolic law

$$G(y)=\frac{G_0h}{h-y}$$

which shows that the modulus increases from G_0 at the surface to an infinite value at a depth h. This simulates in practice a stratum that merges smoothly into a rigid bed at its base and reduces to a homogeneous elastic half-space of constant modulus G_0 when h becomes an infinitely large positive constant. The primary interest is to find out the extent of the dominance of surface shear modulus in such a non-homogeneous stratum.

2. GOVERNING DIFFERENTIAL EQUATIONS

We consider the plane strain problem in which the free surface of the stratum is loaded by a long rigid rectangular strip of semi-width b. Using Cartesian coordinates with axes x, y and z directed laterally, depth-wise and along the longitudinal axis of the strip respectively, we find that

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the elastic equations for the case in which shear modulus varies along y direction alone are:

$$(\lambda + G)\frac{\partial \epsilon}{\partial x} + G\nabla^2 u + \gamma_{xy}\frac{dG}{dy} = 0 \tag{1}$$

$$(\lambda + G)\frac{\partial \epsilon}{\partial y} + G\nabla^2 v + 2\epsilon_y\frac{dG}{dy} + \epsilon\frac{d\lambda}{dy} = 0 \tag{2}$$

where u and v are the components of displacement in the x and y directions, G is the shear modulus, λ is Lame's constant and we record the following relations for dilatation, shear strain and the component of strain in the y-direction respectively.

$$\left.\begin{aligned} \epsilon &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \end{aligned}\right\} \tag{3}$$

If we subtract the partial derivative of eqn (2) with respect to x from that of eqn (1) with respect to y, we find that:

$$(\beta^2 - 1)\frac{\partial \epsilon}{\partial x} g'(y) + g(y)\nabla^2\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + g'(y)\nabla^2 u + \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \epsilon_y}{\partial x} - (\beta^2 - 2)\frac{\partial \epsilon}{\partial x} = 0 \tag{4}$$

where we have introduced

$$\beta^2 = \frac{\lambda + 2G}{G} = \frac{2(1-\nu)}{1-2\nu} \tag{5}$$

in which ν is Poisson's ratio and

$$g(y) = \frac{G(y)}{G'(y)} \tag{6}$$

Equation (4) can be shown to reduce to

$$\nabla^2\omega_z + \frac{g'+1}{g}\frac{\partial \omega_z}{\partial y} - \left[\frac{\beta^2(g'-1)+2}{2g}\right]\frac{\partial \epsilon}{\partial x} = 0 \tag{7}$$

where

$$\omega_z = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] \tag{8}$$

and in which we have used the results

$$\left.\begin{aligned} \nabla^2 u &= \frac{\partial \epsilon}{\partial x} - \frac{2\partial \omega_z}{\partial y} \\ \frac{\partial \gamma_{xy}}{\partial y} - 2\frac{\partial \epsilon_y}{\partial x} &= 2\frac{\partial \omega_z}{\partial y} \end{aligned}\right\} \tag{9}$$

Similar to the above steps, we can show that

$$\nabla^2\epsilon + \frac{g'+1}{g}\frac{\partial \epsilon}{\partial y} + \frac{2(g'+1)}{\beta^2 g}\frac{\partial \omega_z}{\partial x} = 0 \tag{10}$$

in which the following results have been used

$$\left.\begin{aligned} \nabla^2 v &= 2\frac{\partial \omega_z}{\partial x} + \frac{\partial \epsilon}{\partial y} \\ \frac{\partial \gamma_{xy}}{\partial x} + 2\frac{\partial \epsilon_y}{\partial y} &= \nabla^2 v \end{aligned}\right\} \tag{10a}$$

If we now assume a variation of shear modulus given by:

$$g' + 1 = 0 \tag{11}$$

we find, from eqn (6), that

$$G(y) = \frac{G_0 h}{h - y} \tag{12}$$

where G_0 and h are constants.

The governing eqns (7) and (10) then reduce to:

$$\nabla^2 \omega_z = \frac{\partial \epsilon}{\partial x}\left[\frac{1-\beta^2}{g}\right] \tag{13}$$

$$\nabla^2 \epsilon = 0 \tag{14}$$

3. GENERAL SOLUTION OF THE GOVERNING EQUATIONS

The introduction of complex Fourier transform reduces eqns (13) and (14) to

$$\frac{\mathrm{d}^2 \bar{\omega}_z}{\mathrm{d}y^2} - p^2 \bar{\omega}_z = -\frac{ip\bar{\epsilon}(1-\beta^2)}{h - y} \tag{15}$$

$$\frac{\mathrm{d}^2 \bar{\epsilon}}{\mathrm{d}y^2} - p^2 \bar{\epsilon} = 0 \tag{16}$$

If we now introduce the subsidiary variable

$$Y = h - y \tag{17}$$

and the auxiliary function of Poisson's ratio

$$\alpha = 1 - \beta^2 \tag{18}$$

we find that eqns (15) and (16) become

$$\frac{\mathrm{d}^2 \bar{\omega}_z}{\mathrm{d}Y^2} - p^2 \bar{\omega}_z = ip\alpha\frac{\bar{\epsilon}}{Y} \tag{19}$$

$$\frac{\mathrm{d}^2 \bar{\epsilon}}{\mathrm{d}Y^2} - p^2 \bar{\epsilon} = 0 \tag{20}$$

the solution of which can be checked as

$$\bar{\epsilon} = A_1 e^{-pY} + B_1 e^{pY} \tag{21}$$

$$\bar{\omega}_z = A_2 e^{-pY} + B_2 e^{pY} - \frac{i\alpha}{2}[A_1 F(-pY) - B_1 F(pY)] \tag{22}$$

where

$$\left.\begin{aligned} F(-pY) &= e^{pY} Ei(-2pY) - e^{-pY}\log(pY) \\ F(pY) &= e^{-pY} Ei(2pY) - e^{pY}\log(pY) \end{aligned}\right\} \tag{23}$$

in which $Ei(\pm 2pY)$ are the exponential integral functions defined by:

and
$$\left.\begin{aligned} Ei(-x) &= -\int_x^\infty \frac{e^{-t}}{t}\,\mathrm{d}t \\ Ei(x) &= -\int_{-x}^\infty \frac{e^{-t}}{t}\,\mathrm{d}t \end{aligned}\right\} \tag{24}$$

the latter being interpreted as a Cauchy integral.

We record, from the differential properties of $Ei(\pm x)$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} Ei(-x) &= \frac{e^{-x}}{x} \\ \frac{\mathrm{d}}{\mathrm{d}x} Ei(x) &= \frac{e^{x}}{x} \end{aligned} \tag{25}$$

the following relations which are required presently:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}Y}[F(-pY)] &= pH(-pY) \\ \frac{\mathrm{d}}{\mathrm{d}Y}[F(pY)] &= -pH(pY) \end{aligned}\right\} \tag{26}$$

where

$$\left.\begin{aligned} H(-pY) &= e^{pY}Ei(-2pY) + e^{-pY}\log(pY) \\ H(pY) &= e^{-pY}Ei(2pY) + e^{pY}\log(pY) \end{aligned}\right\} \tag{27}$$

and we note that $(\mathrm{d}F/\mathrm{d}y) = -(\mathrm{d}F/\mathrm{d}Y)$ by virtue of eqn (17).

4. STRESS AND DISPLACEMENT TRANSFORMS

By taking the transform of eqn (1) and using the relation

$$2\frac{\partial v}{\partial x} = \gamma_{xy} + 2\omega_z \tag{28}$$

we can show that the transform of the vertical component of displacement is given by:

$$2ip\bar{v} = 2\left[Y\frac{\mathrm{d}\bar{\omega}_z}{\mathrm{d}Y} - \bar{\omega}_z\right] - ip\beta^2 Y\bar{\epsilon} \tag{29}$$

If we now substitute the solutions of $\bar{\epsilon}$ and $\bar{\omega}_z$, we find ultimately that

$$\left.\begin{aligned} 2i\bar{v} &= iA_1\left[\alpha\left\{\frac{F(-pY)}{p} - YH(-pY)\right\} - \beta^2 Ye^{-pY}\right] \\ &\quad - iB_1\left[\alpha\left\{\frac{F(pY)}{p} + YH(pY)\right\} + \beta^2 Ye^{pY}\right] \\ &\quad - 2A_2\left[Y + \frac{1}{p}\right]e^{-pY} + 2B_2\left[Y - \frac{1}{p}\right]e^{pY} \end{aligned}\right\} \tag{30}$$

From the transform of the dilatation, we find

$$ip\bar{u} = -\bar{\epsilon} - \frac{\mathrm{d}\bar{v}}{\mathrm{d}Y} \tag{31}$$

leading ultimately to

$$p\bar{u} = iA_1\left[\frac{\beta^2}{2}e^{-pY}(1+pY) - \frac{\alpha p^Y}{2}F(-pY)\right] + iB_1\left[\frac{\beta^2}{2}e^{pY}(1-pY) + \frac{\alpha pY}{2}F(pY)\right] + pY[A_2e^{-pY} + B_2e^{pY}]. \tag{32}$$

For the shear stress transform,

$$\bar{\tau}_{xy} = \frac{2G_0h}{Y}(ip\bar{v} + \bar{\omega}_z) \tag{33}$$

which results in

$$\bar{\tau}_{xy} = G_0ph\,[iA_1\{\alpha H(-pY) + \beta^2e^{-pY}\} + iB_1\{\alpha H(pY) + \beta^2e^{pY}\} + 2A_2e^{-pY} - 2B_2e^{pY}]. \tag{34}$$

Similarly

$$\bar{\sigma}_y = \frac{G_0h}{Y}[\beta^2\bar{\epsilon} + 2ip\bar{u}] \tag{35}$$

which ultimately gives

$$\bar{\sigma}_y = G_0ph[A_1\{\alpha F(-pY) - \beta^2e^{-pY}\} + B_1\{\beta^2e^{pY} - \alpha F(pY)\} + 2iA_2e^{-pY} + 2iB_2e^{pY}]. \tag{36}$$

5. BOUNDARY CONDITIONS AND THE GOVERNING DUAL INTEGRAL EQUATIONS

On the free surface given by $y = 0$ (or $Y = h$), we assume that the shear stress vanishes throughout and the discontinuous direct stress σ_y will be represented by an unknown distribution $\sigma(x)$ which vanishes outside the strip. At the base of the stratum $y = h$ (or $Y = 0$), we can only assume on physical grounds that the particles which have infinite shear modulus are rigidly attached to the rigid bed. Thus, the four boundary conditions for evaluating the arbitrary functions A_1, B_1, A_2 and B_2 are:

$$\bar{\sigma}_y|_{Y=h} = \overline{\sigma(x)} \tag{37}$$

$$\bar{\tau}_{xy}|_{Y=h} = 0 \tag{38}$$

$$\bar{v}|_{Y=0} = 0 \tag{39}$$

$$\bar{u}|_{Y=0} = 0. \tag{40}$$

The mixed boundary conditions at the surface are:

$$\left.\begin{aligned} v|_{Y=h} &= v_0 \quad |x| < b \\ \sigma_y|_{Y=h} &= 0 \quad |x| > b \end{aligned}\right\} \tag{41}$$

which are required for the formulation of the governing dual integral equations.

We notice from the boundary condition in eqn (40) and the general expression for $\bar{u}$ in eqn (32) that A_1 and B_1 are related in a simple manner according to:

$$A_1 + B_1 = 0. \tag{42}$$

We remark, as an aside here, that the dilatation is seen to be an odd function given by:

$$\bar{\epsilon} = 2B_1 \sinh(pY) \tag{43}$$

which, as expected, vanishes at the base of the stratum where the particles are incompressible being of infinite modulus of rigidity.

Putting now eqn (42) in the remaining boundary conditions and using the appropriate general expressions, we find that A_1, A_2 and B_2 can now be determined from the simultaneous equations:

$$A_1[\alpha\{F(-ph)+F(ph)\}-\beta^2\{e^{-ph}+e^{ph}\}]+2iA_2e^{-ph}+2iB_2e^{ph}=\frac{\overline{\sigma(x)}}{G_0ph} \tag{44}$$

$$iA_1[\alpha\{H(-ph)-H(ph)\}+\beta^2\{e^{-ph}-e^{ph}\}]+2A_2e^{-ph}-2B_2e^{ph}=0 \tag{45}$$

$$-iA_1\alpha F_0+A_2+B_2=0 \tag{46}$$

where

$$F_0=\lim_{Z\to 0}F(\pm pZ)$$
$$=\gamma+\log e^2 \tag{47}$$

in which γ is Euler's constant.

We require the transform of surface vertical component of displacement for the formulation in eqn (41). It can be shown, by virtue of the vanishing of shear stress at the surface according to eqn (38), that

$$2\bar{v}_0p=\frac{\overline{\sigma(x)}}{G_0ph}+\beta^2e^{-ph}A_1-\beta^2e^{ph}B_1 \tag{48}$$

so that, on using eqn (42), we find that

$$\bar{v}_0=\frac{\overline{\sigma(x)}}{2p^2G_0h}+\frac{\beta^2A_1}{p}\cosh(ph) \tag{49}$$

which shows that we require to evaluate only A_1 in eqns (44)–(46).

We find, after some algebra, that

$$A_1=-\frac{2\overline{\sigma(x)}\cosh(ph)}{phG_0\Delta(p)} \tag{50}$$

where

$$\Delta(p)=4[\alpha\{1-T(2ph)\}+\beta^2] \tag{51}$$

in which the function $T(2ph)$, first introduced by Gibson *et al.*[2] in the related problem of a linearly non-homogeneous incompressible stratum, is defined as:

$$T(\eta)=\frac{1}{2}[Ei(\eta)+Ei(-\eta)]-\log\eta-\gamma+1 \tag{52}$$

Substituting for A_1 in eqn (49) and using the mixed boundary conditions at the surface as given by eqns (41), we find from the inversion theorem of complex Fourier transform that the governing dual integral equations for the symmetrical problem of vertical loading of the strip are:

$$\left.\begin{aligned}&\int_0^\infty\frac{\overline{\sigma(x)}\cos(px)}{2p^2G_0h}\left[1-\frac{\cosh^2(ph)}{1+\left(1-\frac{1}{\beta^2}\right)[T(2ph)-1]}\right]dp=v_0(0\leqslant x<b)\\&\int_0^\infty\overline{\sigma(x)}\cos(px)\,dp=0(x>b)\end{aligned}\right\} \tag{53}$$

and, for the antisymmetrical problem of rocking about the longitudinal axis, the cosine term is replaced by the sine and the right hand by $x\psi$ where ψ is the angle of rock.

As a check on the correctness of eqns (53), we consider the limit as h tends to infinity when we find that the half-space equations for a homogeneous medium of constant shear modulus G_0 are recovered.

$$\left.\begin{aligned} \frac{1}{2G_0\left(\frac{1}{\beta^2}-1\right)}\int_0^\infty \frac{\overline{\sigma(x)}}{p}\cos(px)\,\mathrm{d}p &= v_0(0 \leq x < b) \\ \int_0^\infty \overline{\sigma(x)}\cos(px)\,\mathrm{d}p &= 0(x > b). \end{aligned}\right\} \tag{54}$$

In deriving eqns (54), we have used the known asymptotic results:

$$\left.\begin{aligned} \lim_{|x|\to\infty} Ei(\pm x) &\sim \frac{e^{\pm x}}{\pm x}[1+0|x|^{-1}] \\ \lim_{x\to\infty} \cosh x &\sim \frac{e^x}{2}. \end{aligned}\right\} \tag{55}$$

6. THE DOMINANCE OF SURFACE ELASTIC MODULUS FROM AN APPROXIMATE ANALYSIS OF THE DUAL INTEGRAL EQUATIONS

In order to find a first approximation solution to the governing dual integral eqns (53) from which, if necessary, a scheme of successive approximations similar to that in Awojobi and Grootenhuis [5] could be developed, we need to consider first the behaviour of the integrand in the first of eqns (53). The hypothesis that the surface elastic modulus is the dominant parameter requires a comparison of eqns (53) and (54) throughout the range of integration.

We have plotted in Fig. 1 the governing functions:

$$I_1 = \frac{2}{\eta}\left[\frac{\cosh^2(\eta/2)}{1+\kappa[T(\eta)-1]}-1\right] \tag{56}$$

and

$$I_2 = +\frac{1}{\kappa} \tag{57}$$

where

$$\left.\begin{aligned} k(\nu) &= 1-\frac{1}{\beta^2} \\ &= \frac{1}{2(1-\nu)} \end{aligned}\right\} \tag{58}$$

for the extreme values of Poisson's ratio, $\nu = 0$ and $\frac{1}{2}$ and the intermediate value $\frac{1}{4}$.

The behaviour of $T(\eta)$—based on the known power series and asymptotic expansions of $E_i(\pm\eta)$ given by Erdélyi *et al.* [6] for small and large values of η—has been discussed by Gibson *et al.* [2] and their expressions have also been used here.

7. AN ASSESSMENT OF THE EFFECT OF STRATUM DEPTH ON THE DOMINANCE OF SURFACE MODULUS

We can make some assessment of the contribution of stratum depth to the dominance of the surface modulus as obtained from the first approximation consideration of the last section by considering the second approximation in the scheme of successive approximations developed by Awojobi and Grootenhuis [5].

For this purpose, we require to express the first of eqns (53) and (54) in their non-dimensional forms using the variables

$$\eta = 2ph, \bar{x} = \frac{x}{b} \quad \text{and} \quad \bar{h} = \frac{h}{b}$$

so that the transform of the unknown stress distribution $\overline{\sigma(x)}$ becomes $F(\eta)$ and, to a second approximation,

$$F(\eta) \simeq F_{11}(\eta) + F_{22}(\eta) \tag{59}$$

where $F(\eta)$ is obtained from the non-dimensional form of eqns (53), the first of which is

$$\int_0^\infty \frac{F(\eta)}{\eta} \cos\left(\eta \frac{\bar{x}}{2\bar{h}}\right) f(\eta)\,\mathrm{d}\eta = 2\pi G_0 v_0 (0 \leq \bar{x} \leq 1) \tag{53a}$$

and $F_{11}(\eta)$ is obtained from the first of eqns (54)

$$\int_0^\infty \frac{F_{11}(\eta)}{\eta} \cos\left(\eta \frac{\bar{x}}{2\bar{h}}\right) f_3(\eta)\,\mathrm{d}\eta = 2\pi G_0 v_0 (0 \leq \bar{x} < 1) \tag{54a}$$

in which

$$f(\eta) = \frac{2}{\eta}\left\{1 - \frac{\cosh^2\left(\frac{\eta}{2}\right)}{1 + \left(1 - \frac{1}{\beta^2}\right)[T(\eta) - 1]}\right\} \tag{60}$$

$$f_3(\eta) = \frac{1}{\frac{1}{\beta^2} - 1} \tag{61}$$

and the solution of $F_{11}(\eta)$ is known to be

$$F_{11}(\eta) = 2\pi G_0 v_0 \left(\frac{1}{\beta^2} - 1\right) J_0\left(\frac{\eta}{2\bar{h}}\right) \tag{62}$$

It is readily shown from Awojobi and Grootenhuis[5] that $F_{22}(\eta)$ in eqn (59) is to be obtained from the pair of equations

$$\left.\begin{aligned} \int_0^\infty \frac{F_{22}(\eta)}{\eta} \cos\left(\eta \frac{\bar{x}}{2\bar{h}}\right) f_3(\eta)\,\mathrm{d}\eta &= 2\pi G_0 v_0 - \int_0^\infty \frac{F_{11}(\eta)}{\eta} \cos\left(\eta \frac{\bar{x}}{2\bar{h}}\right) f(\eta)\,\mathrm{d}\eta \quad (0 \leq \bar{x} < 1) \\ \int_0^\infty F_{22}(\eta) \cos\left(\eta \frac{\bar{x}}{2\bar{h}}\right) \mathrm{d}\eta &= 0 \quad (\bar{x} > 1). \end{aligned}\right\} \tag{63}$$

As shown in Awojobi and Grootenhuis[5], it is necessary to differentiate the first of eqns (63) with respect to $\bar{x}$ in order to eliminate the singularity at the origin where the integrand on the right becomes infinitely large.

However, it is to be noted that the insertion of eqn (62) into eqn (54a) makes the integral of the latter to diverge because of this singularity at the origin. This difficulty is usually overcome by a well-known technique of finding the *shape* of the deformed surface as a function of $\bar{x}$ by subtracting the infinite constant

$$\int_0^\infty \frac{F_{11}(\eta)}{\eta} \cos\left(\frac{\eta}{2\bar{h}}\right) \mathrm{d}\eta.$$

from the infinite displacement obtained by the insertion of eqn (62) in eqn (54a).

This is tantamount to attaching an origin to the medium and convecting with it in a manner similar to the technique adopted by Awojobi and Gibson (Ref. [4] p. 291, eqn 5.12).

The dominant effect of stratum depth in comparison with surface shear modulus can now be obtained from the right-hand side of the first of eqns (63) as

$$2\pi G_0 v_0[1-\epsilon_{11}(\bar{x},\bar{h})] \tag{64}$$

where the depth effect

$$\epsilon_{11}(\bar{x},\bar{h}) = \int_0^{\bar{x}} \left[\frac{1}{2\bar{h}}\int_0^{\infty} F_{11}(\eta)\sin\left(\eta\frac{\bar{x}}{2\bar{h}}\right)f(\eta)\,\mathrm{d}\eta\right]\mathrm{d}\bar{x} \quad (0<\bar{x}<1), \tag{65}$$

is to be compared with the surface modulus effect given by the first term, unity, inside the brackets of eqn (64).

In order to evaluate (65) we split the interval of integration into three parts corresponding to

$$\left.\begin{aligned} f(\eta) &\sim f_1(\eta) = -\frac{\eta}{2} \quad (0<\eta<\eta_1)\\ &\sim f_2(\eta) = \frac{2}{\eta} + \frac{1}{\frac{1}{\beta^2}-1} \quad (\eta_1<\eta<\eta_2)\\ &\sim f_3(\eta) = \frac{1}{\frac{1}{\beta^2}-1} \quad (\eta>\eta_2) \end{aligned}\right\} \tag{66}$$

where η_1 is small and chosen such that

$$f_1(\eta) = f_2(\eta) \tag{67}$$

and η_2 is large enough to make

$$f_2(\eta) \sim f_3(\eta) \tag{68}$$

In eqns (66) we have used the known results

$$E_i(\pm\eta) \sim \gamma + \log(\eta) + 0(\eta) \quad \text{for small values of } \eta$$

and

$$E_i(\pm\eta) \sim \frac{e\pm\eta}{\pm\eta}[1+0|\eta|^{-1}] \quad \text{for large values of } \eta.$$

Substituting for $F_{11}(\eta)$ in eqn (62) and using the power series and asymptotic expansions of the Bessel Function as appropriate, we find ultimately that eqn (64) evaluates to

$$1-\epsilon_{11}(\bar{x},\bar{h}) \sim 1 - \frac{\bar{x}^2}{16\bar{h}^2}\left[1+\frac{1}{3}\left(\frac{1}{\beta^2}-1\right)\right] + 0\left(\frac{1}{\bar{h}^3}\right) \quad (0\leqslant\bar{x}<1). \tag{69}$$

and, relating β^2 to Poisson's ratio ν as in eqn (5), we find

$$1-\epsilon_{11}(\bar{x},\bar{h}) \sim 1 - \frac{\bar{x}^2}{16\bar{h}^2}\left[1-\frac{1}{6(1-\nu)}\right] \quad (0\leqslant\bar{x}<1). \tag{70}$$

Thus, in the worst case $\bar{x}=1$, the dominant effect of stratum depth compared with surface shear modulus effect is shown in Table 1 for the extreme and intermediate values of Poisson's ratio and for depth ratios $\bar{h} = 1.5, 2, 3, 4, 5, 10$. This effect has been calculated from the second term of eqn (70) as a percentage of the first from which we conclude that surface shear modulus is by far the dominant parameter since stratum depth contributes less than 1% for all depth ratios $\bar{h}>2$.

It is interesting to note that, using a numerical procedure for evaluating the integrals, the same

Table 1. Effect of stratum depth as a percentage of surface shear modulus effect

ν \ $\bar{h}$	1.3	2	3	4	5	10
0	−2.32	−1.30	−0.58	−0.33	−0.21	−0.052
1/4	−2.16	−1.21	−0.54	−0.30	−0.19	−0.049
1/2	−1.85	−1.04	−0.46	−0.26	−0.17	−0.042

conclusion—that the effect of stratum depth is negligible for $\bar{h} > 2$—is reached by Gibson *et al.* [2] who considered the similar problem of a stratum with base fixed to a rigid bed but with shear modulus varying linearly with depth.

It should be noted that $\epsilon_{11}(\bar{x})$ is the contribution of $F_{22}(\eta)$ to the vertical displacement v_0 and, therefore, represents the correct basis of comparison with Gibson *et al.* (2) who give the numerical results for settlement of the rectangular strip on the assumption of uniform pressure loading over the strip.

DISCUSSION AND CONCLUSIONS

The comparison of the governing functions in Fig. 1 between the exact formulation of the hyperbolic variation case and the homogeneous elastic half-space solution proves that, as a first approximation, the surface elastic properties of a non-homogeneous medium are the dominant parameters. The major assumption is that the lateral variation of these elastic properties is negligible. This is reasonable because the variation of soil properties over an area comparable with the dimensions of the base of structures can be regarded small compared with depth variation where the increase at a depth equal to the base average width can be as large as the surface modulus itself.

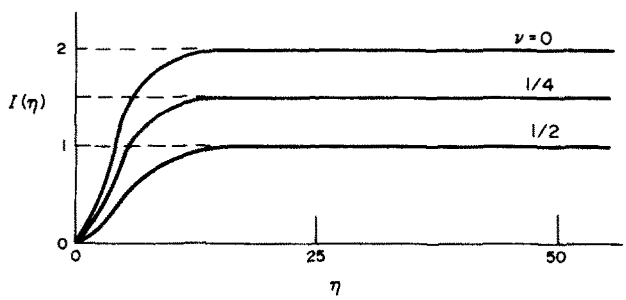

Fig. 1. Comparison between exact governing function (full lines) and the approximation (broken lines) due to surface elastic modulus only.

The dominance of surface modulus becomes more pronounced when the body is subjected to any motion since the stiffening of the static stress distribution will be relaxed by virtue of apparent increase in the inertia of the body due to motion of the surrounding soil. This compensating effect has been discussed in a previous work, Awojobi[7].

Finally, the hyperbolic variation of shear modulus with depth is indeed very practical as it shows a smooth transition of the sub-soil into a rigid bed. In this regard, it is superior to a linearly increasing modulus for a stratum since this assumes the existence of a plane surface at some depth h where there is a sharp change from non-rigid to infinitely rigid particles of soil.

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